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Some Results on Time Scale & Differential Equations Involving Generalised Fractional Derivative via Fractional Integral Transforms

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Introduction:

Fractional Calculus:

Differential equations is a major branch of mathematics since their inauguration in 17th century. A gradual attempt of solving physical problems led to mathematical models involving equations in which function and its derivative play important roles. Differential equations began with Leibniz, the Bernoulli brothers and others from 1680, just after Newton's "Fluxional Equation" in the 1670. Most 18th century development consolidated Leibniz tradition. Further new figures are added, especially by Euler, Daniel Bernoulli, Lagrange and Laplace. In 19th century, general theory was enriched by development of the understanding of general and particular solutions of existence theorems. More types of equations and their solutions appeared in 20th century. General theory was influenced by arrival of set theory in mathematical analysis, which further includes topological aspects. The concept of fractional derivate was introduced after 1695, as a generalisation of integral derivative. It was first mentioned in Leibniz letter to L'Hospital, where the idea of semi-derivative was suggested. Further fractional derivative was built on foundations by many famous mathematics. After L'Hospital's and Leibniz's fractional calculus was primarily a study reserved for the Best minds in mathematics, Fourier, Euler, Laplace are among the many that deal with fractional calculus. Subsequent references to fractional derivatives were made by Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Weyl in 1919, and others. Lacroix develop the formula of nth derivative of $y = x^m$ with m being positive integer,

$$D_x^n y = \frac{d^n}{dx^n}\left(x^m\right) = \frac{m!}{(m-n)!}x^{m-n},\ m \geq n.$$

Replacing factorial symbol by Gamma function, he developed the formula for fractional derivative of a power function,

$$D_x^\alpha x^\beta = \frac{\Gamma\left(\beta+1\right)}{\Gamma\left(\beta-\alpha+1\right)}x^{\beta-\alpha},$$

where α & β are fractional numbers and where the gamma function $\Gamma(z)$ is defined for $z > 0$ as :

$$\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx.$$

There are many definitions on fractional calculus most famous of these definitions that have been popularized in the world of fractional calculus are the Riemann-Liouville and Grunwald-Letnikov definition.Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However, in the past 100 years, it is an interesting topic in engineering and scientific application. Caputo reformulated the more classic definition of the Riemann-Liouville fractional derivative in order to use integer-order initial conditions to solve fractional-order differential equations. As recently as 1996, Kolwankar reformulated once again, the Riemann-Liouville fractional derivative in order to differentiate nowhere differentiable fractal functions. The generalization of Riemann-Liouville and Caputo fractional derivatives was introduced by R. Hilfer in 1999. A significant development and interest has been shown by many researchers. Vanterler et al. interpolated Hilfer Fractional Derivative (HFD) and ψ-fractional derivative is called as ψ-HFD. This fractional derivative is different from other classical fractional derivatives because the kernel is in terms of function.

Definition 1. Riemann Liouville Fractional Derivative:

For $\alpha > 0$, the integral

$$ {}_aD_x^{\alpha} f(x) = \frac{d^n}{dx^n}\left[{}_aD_x^{n-\alpha} f(x)\right] = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)dt, $$

is called as the Riemann Liouville Fractional derivative of order α.

Definition 2. Riemann Liouville Fractional Integral:

The Riemann Liouville fractional integral of order α, $(n-1 < \alpha \leq n)$ is defined as,

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt. $$

Definition 3. Caputo's Fractional Derivative:

The Caputo's Fractional derivative of $f(x)$ of order $\alpha > 0$ is defined as,

$$ {}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t)dt. $$

Defintion 4. The left sided Hilfer fractional derivative of order α:

The left sided Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ of function $f(x)$ is defined as,

$$D_{a+}^{\alpha,\,\beta} f(x) = I_{a+}^{\beta(1-\alpha)} D \left(I_{a+}^{(1-\alpha)(1-\beta)} f \right)(x).$$

Fractional Integral Transform:

Fractional integral transforms happens to be most powerful tool, which is extension of traditional integral transforms to non-integral order. It has gain attention of many researcher in the field of mathematics, physics, engineering, life sciences, and other areas of science. As we know that the classical integral transform like Laplace, Fourier, Hankel can be effectively used to solve certain types of differential equations, likewise the fractional differential equations arises as a result of mathematical modelling of real life. Problems can be solved by fractional, integral, transform in both analytical and numerical approach. The integral transforms translates a D.E. into an algebraic equations.

The mathematician Bernhard Riemann who introduced the concept of fractional integral transform in middle of the 19th century. Later on Emst Lindelöf and other contributed to the development of fractional integral transform significantly to each integral transform.
Integral transform methods are used to solve the differential and integral equations and to study special functions. The techniques of integral transforms effectively use in treating the linear operators of fractional derivative and fractional integration in a way which can avoid excessive mathematical meticulousness.
Integral transform of a function $f(t)$ is defined as,

$$F(s) = \int_{t_1}^{t_2} f(t) K(t,\, s) dt,$$

where $K(t,\, s)$ is called as the Kernal of the integral transform. For different values of $K(t,\, s)$ we will get different integral transforms. The inverse integral transform is defined as,

$$f(t) = \int_{s_1}^{s_2} F(s) K^{-1}(s,\, t) ds.$$

Some Integral Transforms:

Let $f(t)$ be continues on an interval $[0,\infty)$ which is exponential order i.e. for $c \in \mathbb{R}$ & $t > 0$, Sup $\left|f(t)e^{-ct}\right| < \infty$, then integral transform of $f(t)$ will exists.

Formulae for some integral transform of fractional order integral and fractional derivatives are:

1. **Laplace Transform** $L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \ \ s > 0$

   i) $L\left\{{}_0^C D_t^\sigma f(t)\right\} = s^\sigma F(s) - \sum_{n=0}^{k-1} s^{\sigma-n-1} D_t^n f(0)$

   ii) $L\left\{{}_0^{RL} D_t^\sigma f(t)\right\} = s^\sigma F(s) - \sum_{n=0}^{k-1} s^\sigma D_t^{\sigma-n-1} f(0)$

   iii) $L\left\{{}_0^{RL} D_t^{-\sigma} f(t)\right\} = s^{-\sigma} F(s)$

2. **Sumudu Transform** $S\left\{f(t)\right\} = \int_0^\infty f(st)e^{-t}dt = G(s), \ \ s > 0$

   i) $S\left\{{}_0^C D_t^\sigma f(t)\right\} = s^{-\sigma} G(s) - \sum_{n=0}^{k-1} s^{n-1} D_t^n f(0)$

   ii) $S\left\{{}_0^{RL} D_t^\sigma f(t)\right\} = s^{-\sigma} G(s) - \sum_{n=0}^{k-1} s^{-n-1} D_t^{\sigma-n-1} f(0)$

   iii) $S\left\{{}_0^{RL} D_t^{-\sigma} f(t)\right\} = s^\sigma G(s)$

3. **Elzaki Transform** $E\{f(t)\} = s\int_0^\infty f(t)e^{-t/s}dt = E(s), \ \ s > 0$

   i) $E\left\{{}_0^C D_t^\sigma f(t)\right\} = s^{-\sigma} E(s) - \sum_{n=0}^{k-1} s^{n-\sigma+2} D_t^n f(0)$

   ii) $E\left\{{}_0^{RL} D_t^\sigma f(t)\right\} = s^{-\sigma} E(s) - \sum_{n=0}^{k-1} s^{1-n} D_t^{\sigma-n-1} f(0)$

   iii) $E\left\{{}_0^{RL} D_t^{-\sigma} f(t)\right\} = s^\sigma E(s)$

4. Shehu Transform $H\{f(t)\} = \int_0^\infty f(t)e^{-st/u}dt = V(s, u) \quad s, u > 0$

i) $H\left\{{}_0^c D_t^\sigma f(t)\right\} = \frac{s^\sigma}{u^\sigma}V(s, u) - \sum_{n=0}^{k-1} \frac{s^{\sigma-n-1}}{u^{\sigma-n-1}} D_t^n f(0)$

ii) $H\left\{{}_0^{RL} D_t^\sigma f(t)\right\} = \frac{s^\sigma}{u^\sigma}V(s, u) - \sum_{n=0}^{k-1} \frac{s^n}{u^n} D_t^{\sigma-n-1} f(0)$

iii) $H\left\{{}_0^{RL} D_t^{-\sigma} f(t)\right\} = \frac{u^\sigma}{s^\sigma}V(s, u)$

Time Scale Calculus:

The time scale calculus was introduced by Stephen Hilger in 1998. The theory of Time scale calculus allows to prove results for differential equations and difference equations. It is unification of theory of difference equations. It is very useful in a framework in which discrete and continuous data are required. This calculus has a new definition of derivative. If we differentiate a function defined over real number, then this new definition is equivalent to standard differentiation. And it is equivalent to forward operator if we differentiate a function defined on integers. Time scale $\mathbb{T}$ is closed subset of $\mathbb{R}$. Laplace transform can be defined for function on time scale, which can be used to solve dynamic equations on time scale.

Definition 5:

A function $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$ if,

$$f^\Delta(t) = \lim_{s \to t} \frac{f\left(\sigma(t)\right) - f(s)}{\sigma(t) - s}, \text{ where } \sigma(t) \text{ is jump operator on } \mathbb{T}.$$

Definition 6:

Let $f : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t), \forall\ t \in \mathbb{T}^k$, then F is called antiderivative of f and $\int_a^b f(\tau)\Delta\tau = F(b) - F(b)$.

If $\mathbb{T} = \mathbb{R}$, then derivative and integral coincide with usual derivative and integral respectively.

Literature Review:

Time scale calculus was first introduced by Stefan Hilger in 1998 [26]. Ravi P. Agrawal and Martin Bohner [25] Developed some basic tools of calculus on time scale in 1999. Martin Bohner and Allen Peterson introduced transform and Z-transform on time scale which is equal to Laplace transform (when $\mathbb{T} = \mathbb{R}$) and usual Z-Transform (when $\mathbb{T} = \mathbb{Z}$) [18]. In 2010, N. R. O. Bastos et. al. [23] introduced fractional order integral and fractional order derivative using inverse Laplace transform on time scales. Later on H. A. Agwa et. al. defined Sumudu Transform on time scale and studied it's properties [14]. Da-Fang Zhao and Xue-Xiao You in [11], introduced delta fractional derivative on time scales in 2016. In 2018, D. Ucar et. al. [10], used delta Riemann-Liouville fractional integrals to establish some new integral inequality for Chebyshev functional. New properties of time scale fractional operators was studied by Amin B. Cherif and Fatima Z. Ladrani [1]. In 2021 Ganie and Renu Jain [16] introduced Sumudu transform on discrete time scale. Further T. G. Thange and S. Chhatraband studied generalisation of Shehu transform on time scale in 2022 [29]. In 2024 T. G. Thange and S. Chhatraband introduced Laplace and α-Laplace transform on time scale [28, 30].

New definition of fractional derivative was introduced by Hilfer [15]. Later on many definitions of Hilfer type derivative where introduced like ψ-Hilfer, Hilfer-Hadamard, Hilfer-Katugampola, Hilfer-Prabhakar etc. A. Salamooni, D. D. Pawar [7], studied Couchy-Type problem of differential equations involving Hilfer-Hadamard type derivative. A. F. S. Wagdi and D. D. Pawar [6], proved existence and uniqueness of generalised Cauchy-Type problem. Further A. F. S. Wagdi and D. D. Pawar in [3, 4], gave solution of fractional kinetic equation for Hadamard-Type fractional integral via Mellin transform and fractional kinetic equations involving generalised V-Functions via Laplace transform. Hilfer-Prabhakar fractional derivative became popular as it can incorporate various integral transforms like Laplace, Sumudu, Shehu, Elzaki etc. [12, 24]. Mohmad Khalid, S. Alha [21] applied Sawi Transform on Hilfer-Prabhakar fractional derivative and studied some Cauchy type fractional differential equation.

Motivation for Research:

The classical integral transform proved to be effective tool to solve differential and integral equations, in various fields of Science & Engineering. Many real life problems with non local behaviours can be addressed by using fractional calculus and hence fractional integral transforms becomes essential and vital tool to tackle with such problems. Integral transforms applied on time scale provides unified approach for problems involving continuous and discrete framework. ψ

-Hilfer fractional derivative is most general definition of fractional derivative as it can interpolates between Caputo and Riemann-Liouville derivative.

This effective background of time scale and significance of Hilfer fractional derivative and usefulness of fractional integral transform, motivated us for this study.

Objectives:

1. To extend current results of integral transform on time scales to different integral transforms on time scales.
2. To apply double integral transform such as Laplace-ARA and Double Laplace transform on time scale and study their properties.
3. To study Euler-Cauchy equations with respect to integral transforms in the time-scale domain.
4. To solve dynamical equations by applying integral transforms.
5. To explore the applicability of ARA transform on Hilfer-Prabhakar fractional derivatives.
6. To solve Cauchy-type fractional differential equations involving Hilfer-type derivatives by applying ARA transform.
7. To study the different inequalities arising in dynamical systems via integral transforms in time scale.

Plan of Research:

1. **First Year Plan:** Collection of relevant data by investigating current developments in various fields of fractional calculus and fractional integral transforms. Study of time, scale framework, thoroughly and explore possibility of new integral transform on Time scale.
2. **Second Year Plan:** To study the Hilfer type derivatives recently used in various problems and try to solve the problems using other integral transforms on Hilfer derivatives and investigate the usefulness. Investigation of certain properties on fractional integral when applied to Hilfer type derivatives and on Time Scales.
3. **Third Year Plan:** Try to extend the results of current research to the integro-differential equations involving Hilfer type fractional derivatives. To explore possibility of solution of dynamical systems using selected integral transform, finally to submit dissertation.

Methodology:

1. Thorough investigation of current research papers on time scale and Hilfer type fractional derivatives.
2. Books and research articles on fractional calculus and time scale framework.
3. Discussion with the people working in the same field.

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